Optimization Based Control, Reduction, and Identification of Port-Hamiltonian Systems

Paul Schwerdtner

Brig, May 24

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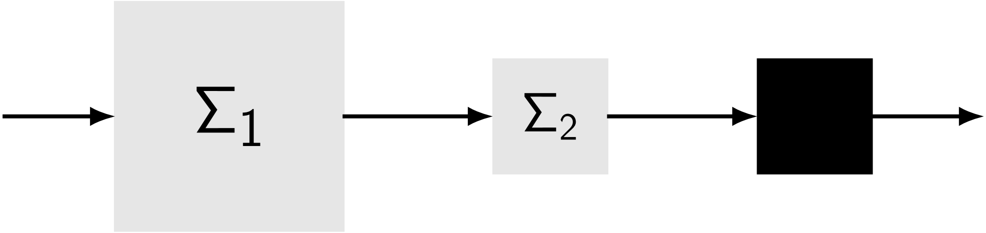
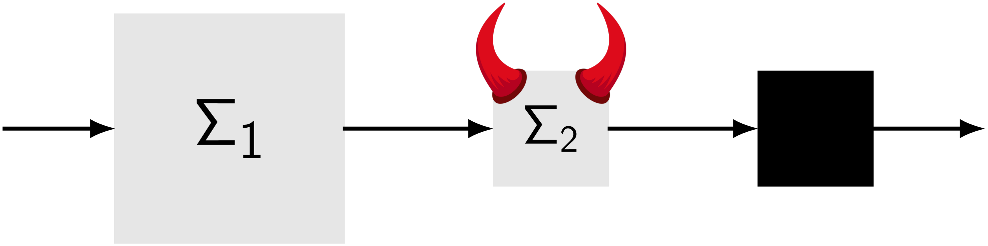

Σ_1
Σ_2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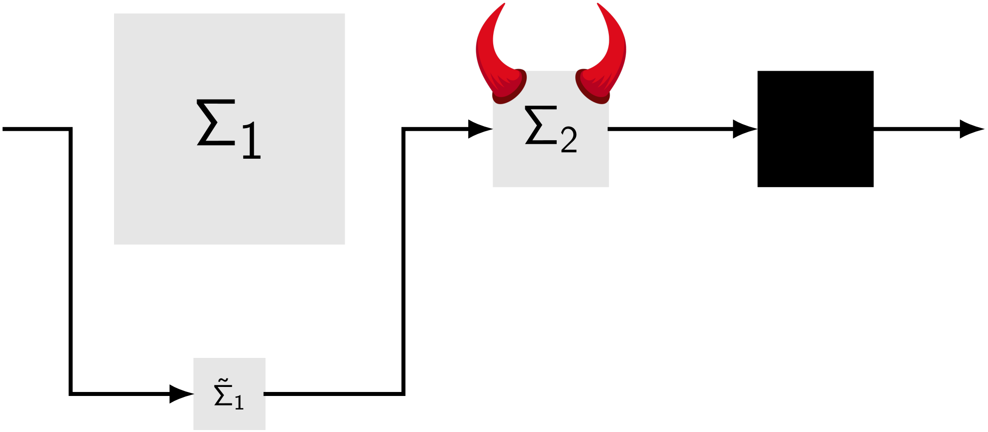
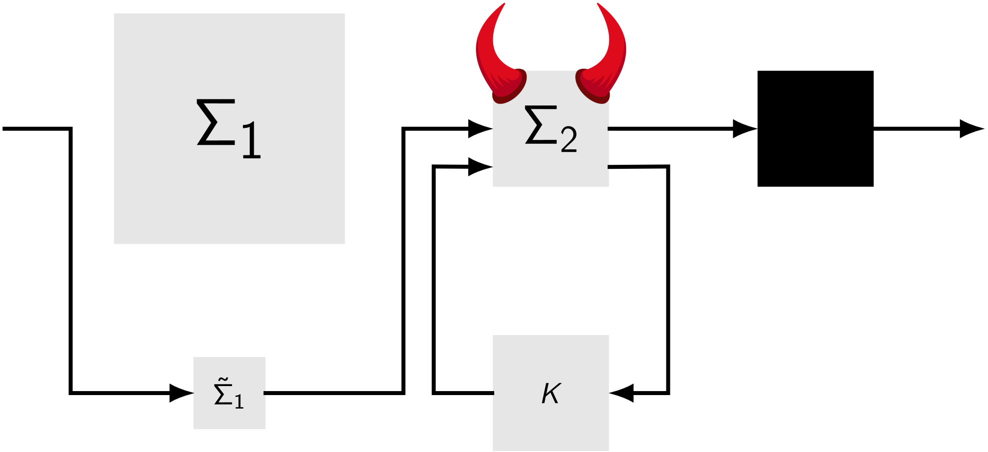
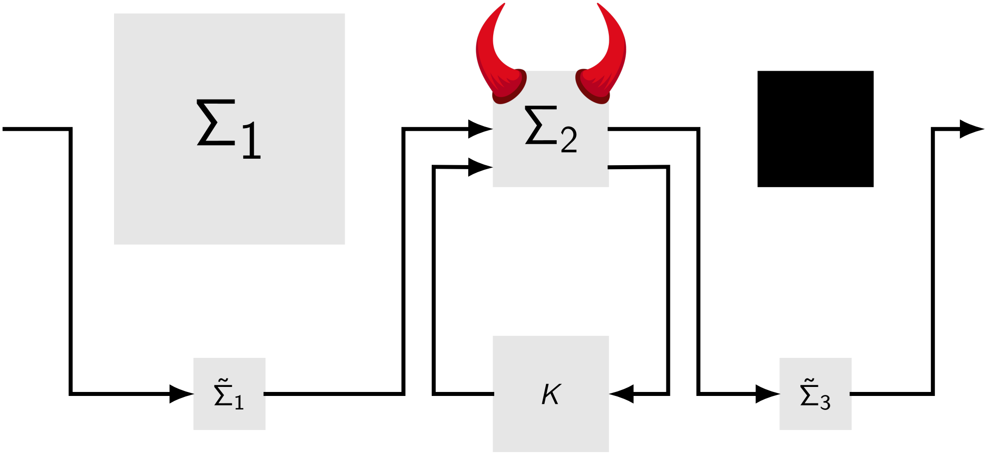


Σ_1
Σ_2
$\tilde{\Sigma}_1$


Σ_1
Σ_2
$\tilde{\Sigma}_1$
K



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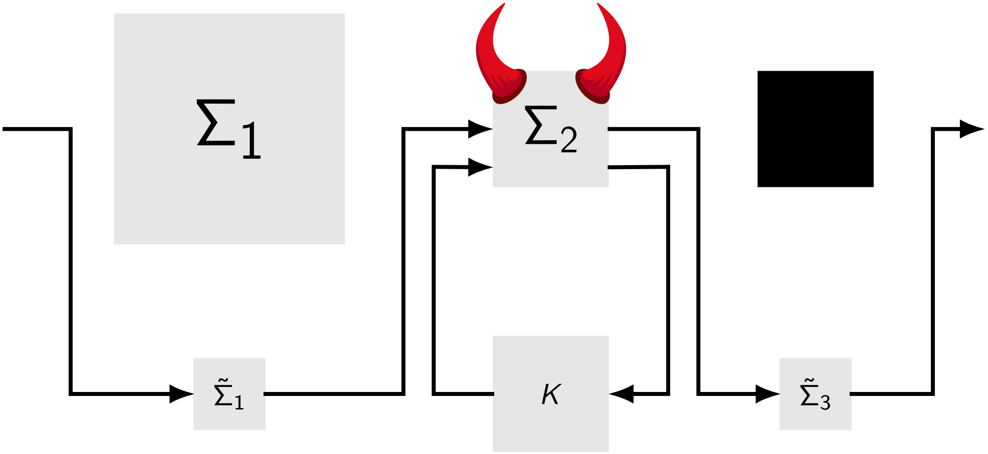
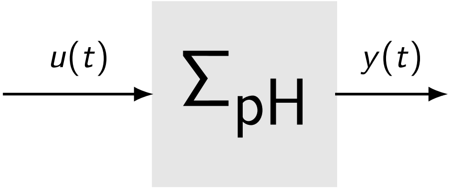
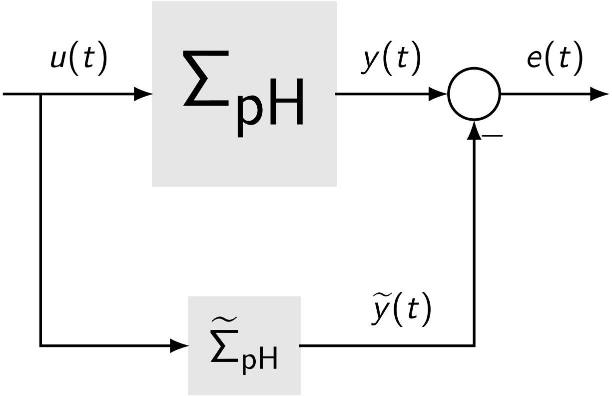


SOBMOR

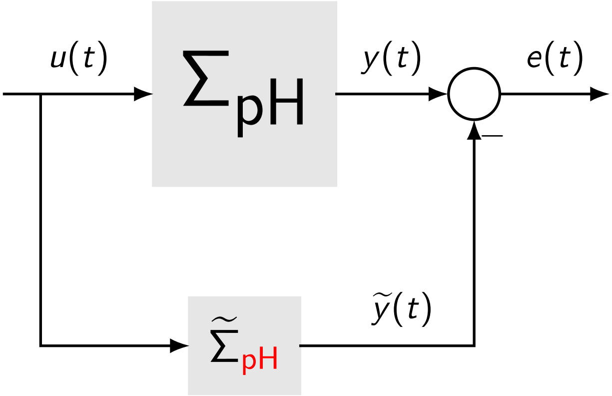
Structure Preserving Model Order Reduction



Structure Preserving Model Order Reduction



Structure Preserving Model Order Reduction



Port-Hamiltonian Systems

$$u(t) \longrightarrow \quad \Sigma_{\mathrm{pH}} : \begin{cases} \dot{x}(t) = (J - R)x(t) + (B - P)u(t) \\ y(t) = (B + P)^{\mathsf{T}} x(t) + (S - N)u(t) \end{cases} \quad \longrightarrow y(t)$$

Dimensions: $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$. We have $n \gg m$.

Constraints:

(i) J and N are skew-symmetric.

(ii) $W_{\mathrm{P}} := \begin{bmatrix} R & P \\ P^{\mathsf{T}} & S \end{bmatrix} \geq 0$.

Error Estimate

$$u(t) \longrightarrow \boxed{\Sigma_{\mathrm{pH}} : \begin{cases} \dot{x}(t) = (J - R)x(t) + (B - P)u(t) \\ y(t) = (B + P)^{\mathsf{T}} x(t) + (S - N)u(t) \end{cases}} \longrightarrow y(t)$$

Error Estimate

$$U(s) \longrightarrow \boxed{\underbrace{(B+P)^{\mathsf{T}}(sI-(J-R))^{-1}(B-P)+S-N}_{H(s)}} \longrightarrow Y(s)$$

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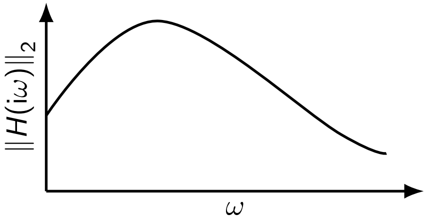
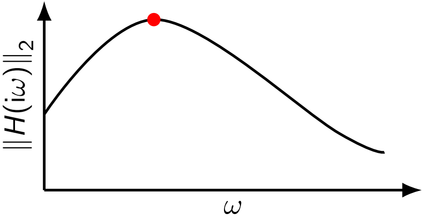


Error Estimate

$$U(s) \longrightarrow \underbrace{(B+P)^{\mathsf{T}}(sI-(J-R))^{-1}(B-P)+S-N}_{H(s)} \longrightarrow Y(s)$$



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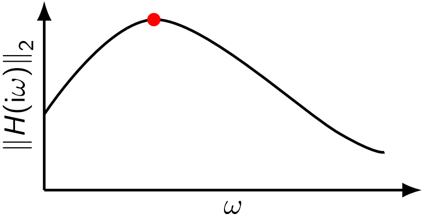
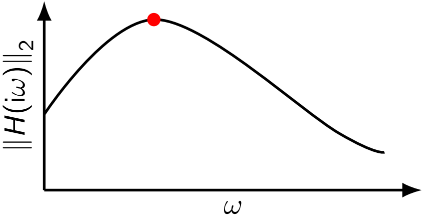


$$\|y\|_{\mathcal{L}_2} \leq \|H\|_{\mathcal{H}_\infty} \|u\|_{\mathcal{L}_2}$$

Error Estimate

$$U(s) \longrightarrow \boxed{\underbrace{(B+P)^{\mathsf{T}}(sI-(J-R))^{-1}(B-P)+S-N}_{H(s)}} \longrightarrow Y(s)$$



$$\|y\|_{\mathcal{L}_2} \leq \|H\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{L}_2}$$

$$\|e\|_{\mathcal{L}_2} \leq \|H-\widetilde{H}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{L}_2}$$

Parametrized port-Hamiltonian ROM

$$\Sigma_{\mathsf{pH}} : \begin{cases} \dot{x}(t) = (J - R)x(t) + (B - P)u(t) \\ y(t) = (B + P)^{\mathsf{T}} x(t) + (S - N)u(t) \end{cases}$$

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$$\Sigma_{\mathrm{pH}}(\theta) : \begin{cases} \dot{x}(t) = (J(\theta) - R(\theta))x(t) + (B(\theta) - P(\theta))u(t) \\ y(t) = (B(\theta) + P(\theta))^{\mathsf{T}} x(t) + (S(\theta) - N(\theta))u(t) \end{cases}$$

Parametrized port-Hamiltonian ROM

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Consider $vtu : \mathbb{R}^q \to \mathbb{R}^{n \times n}$,

$$v \mapsto \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ 0 & v_{n+1} & \cdots & v_{2n-1} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & v_{n(n+1)/2} \end{bmatrix}$$

Parametrized port-Hamiltonian ROM

$$\Sigma_{\mathrm{pH}}: \begin{cases} \dot{x}(t) = (J - R)x(t) + (B - P)u(t) \\ y(t) = (B + P)^{\mathsf{T}} x(t) + (S - N)u(t) \end{cases}$$

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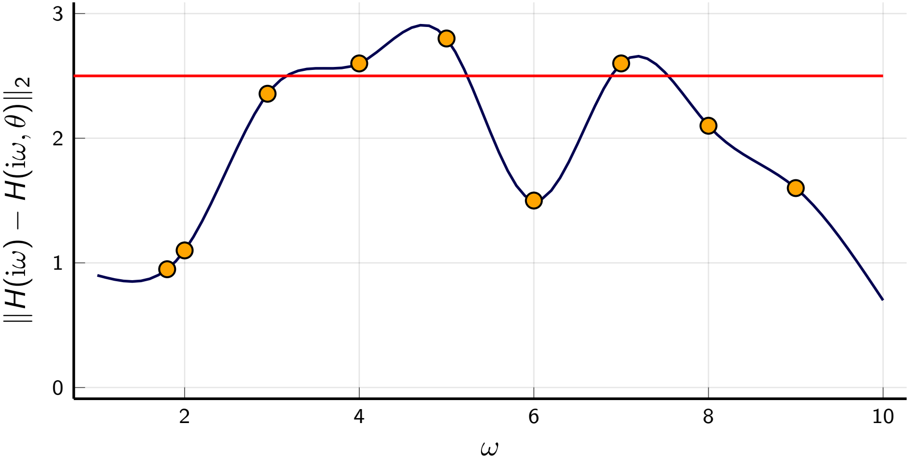
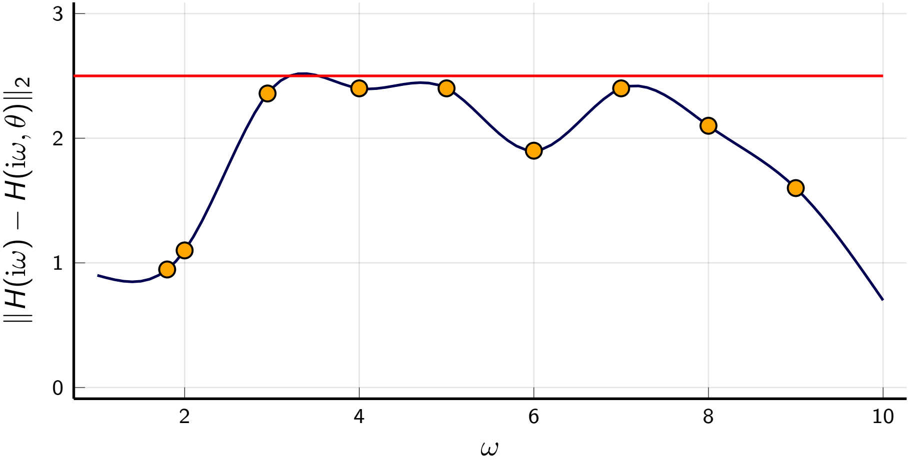
- $J(\theta) = vtu(\theta_J)^{\mathsf{T}} - vtu(\theta_J)$
- $W(\theta) = \begin{bmatrix} R(\theta) & P(\theta) \\ P(\theta)^{\mathsf{T}} & S(\theta) \end{bmatrix} = vtu(\theta_W)^{\mathsf{T}} vtu(\theta_W)$

Algorithm Idea



Algorithm Idea



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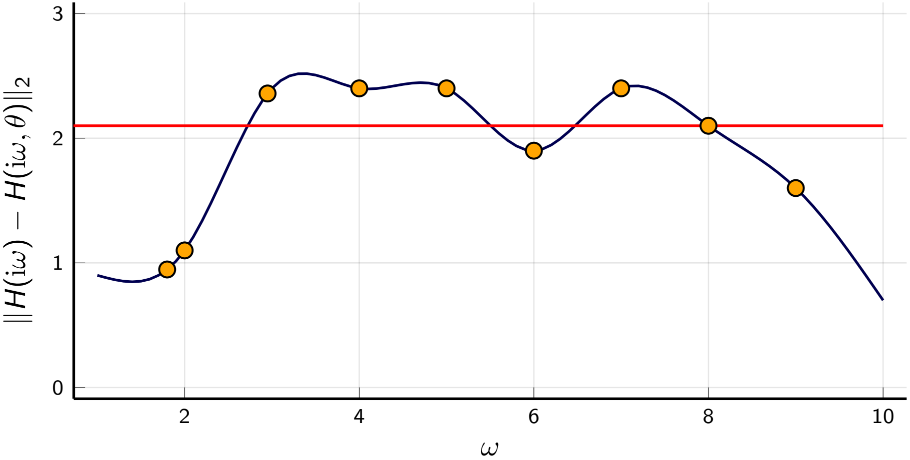

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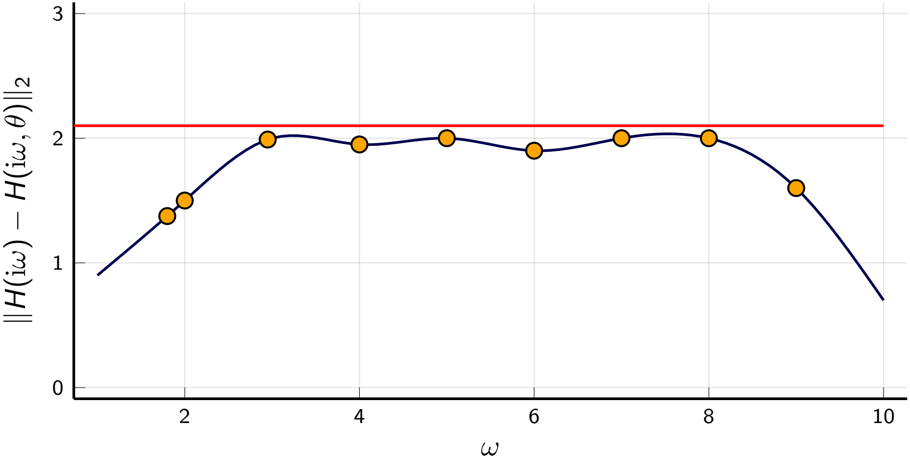

Objective Function

$$L(\gamma, H, \widetilde{H}(\cdot,\theta), S) = \frac{1}{\gamma} \sum_{s_i \in S} \left(\left(\| H(s_i) - \widetilde{H}(s_i,\theta) \|_2 - \gamma \right)_+ \right)^2,$$

where $S \subset \mathrm{i}\mathbb{R}$, $\gamma \in \mathbb{R}^+$, and

$$(\,\cdot\,)_+ : \mathbb{R} \to \mathbb{R}^+, x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

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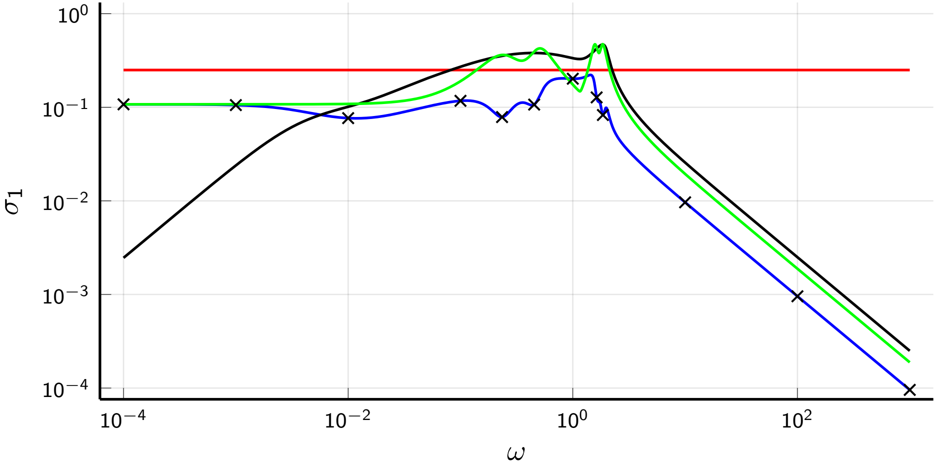
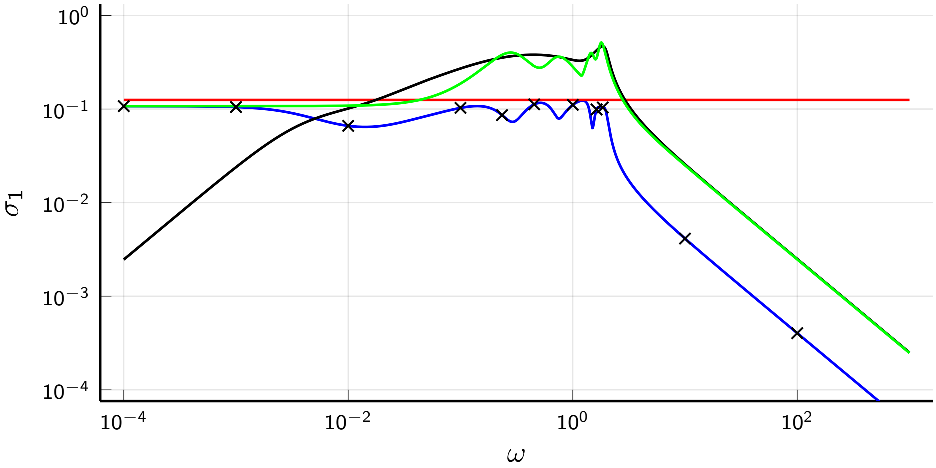
Benefits

- No need to compute the $\mathcal{H}_\infty$-norm of $H - \widetilde{H}(\cdot,\theta)$.
- Gradient contains information on all samples s_i where $\|H(s_i) - \widetilde{H}(s_i)\|_2 > \gamma$.

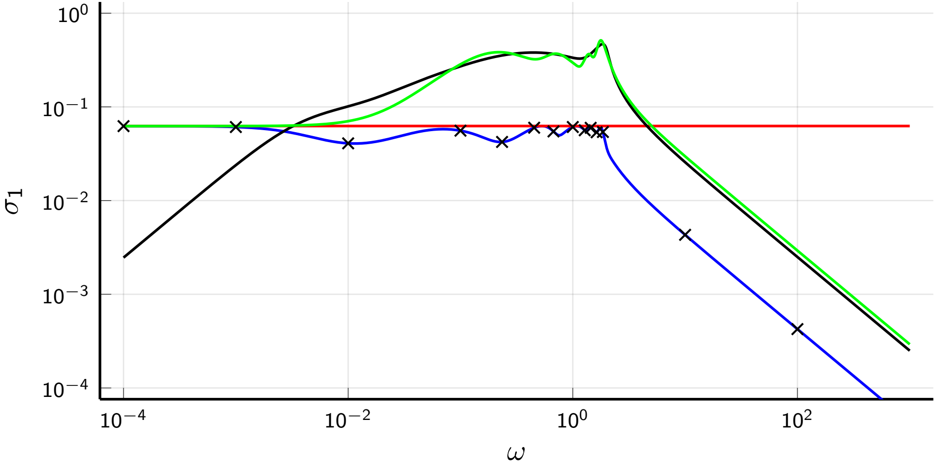
Example



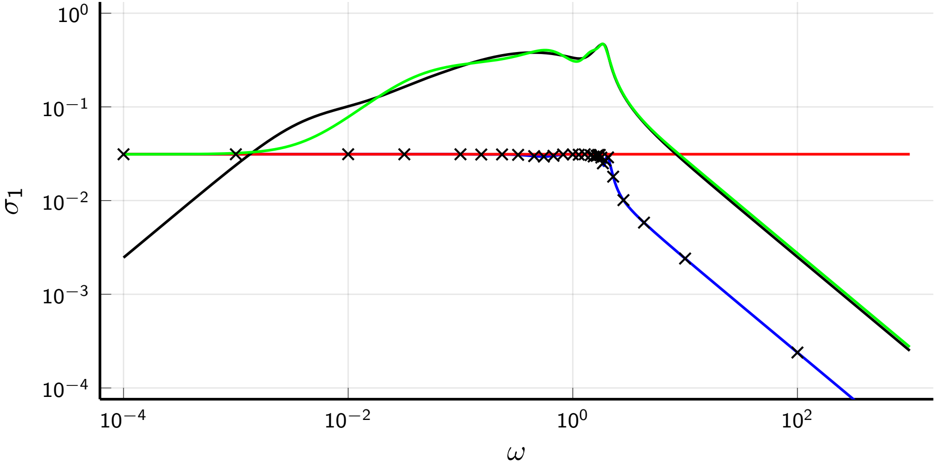
Example



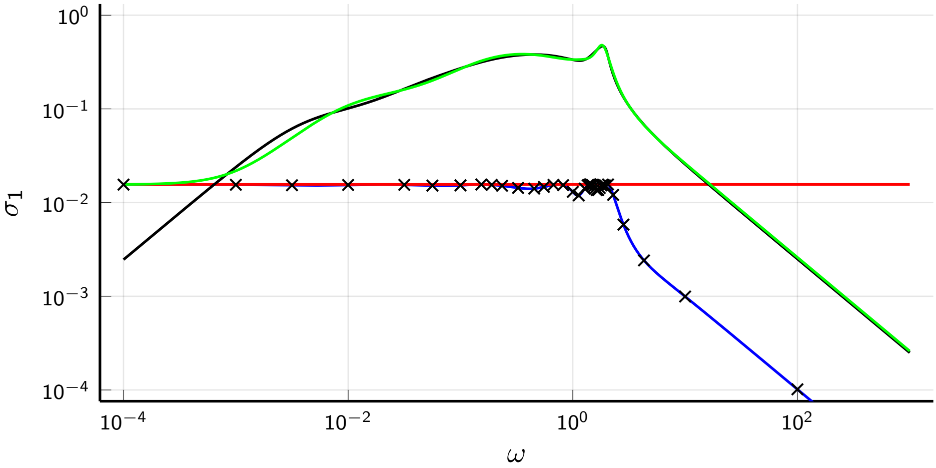
Example



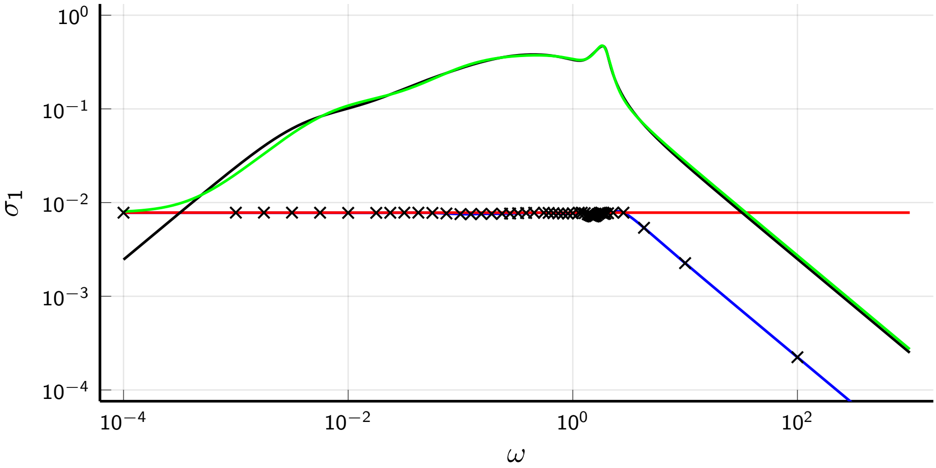
Example



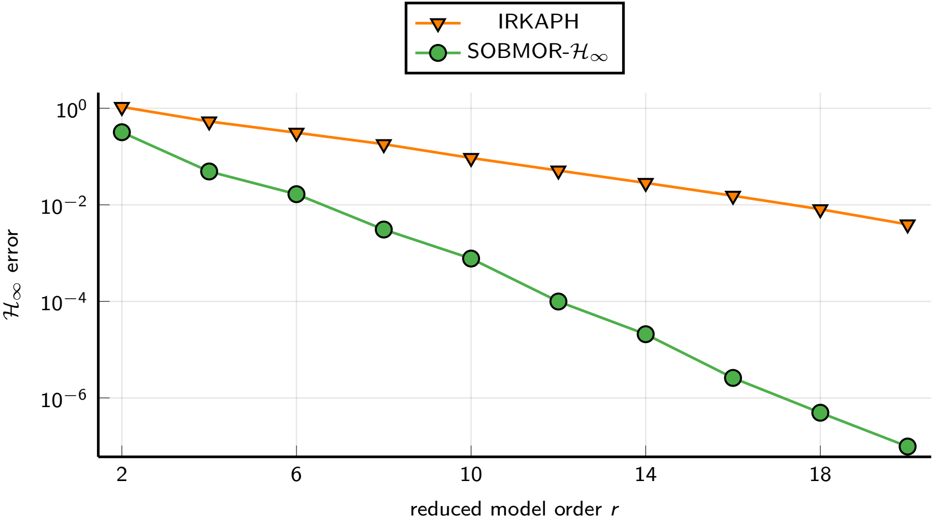
Example



Example

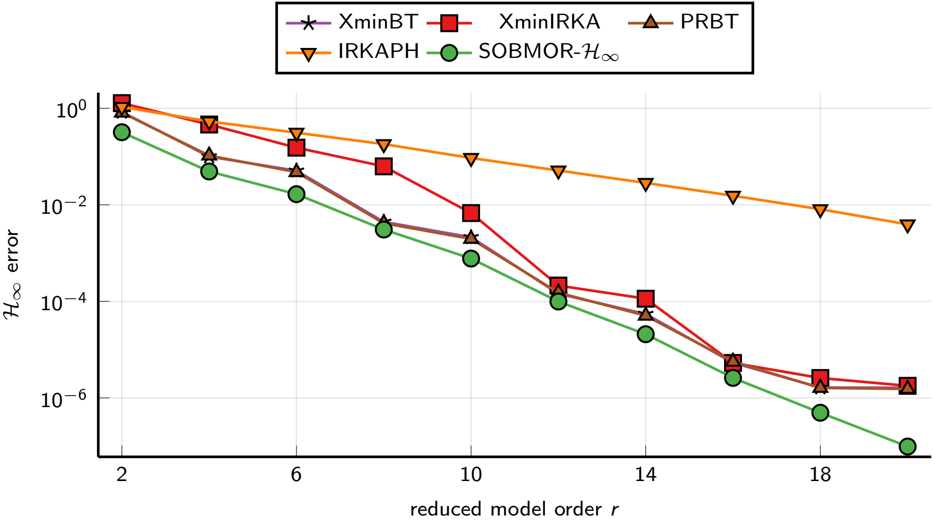


Results



IRKAPH
SOBMOR-$\mathcal{H}_\infty$
10^0
10^{-2}
10^{-4}
10^{-6}
$\mathcal{H}_\infty$ error
2
6
10
14
18
reduced model order r

Results



XminBT
XminIRKA
PRBT
IRKAPH
SOBMOR-$\mathcal{H}_\infty$
10^0
10^{-2}
10^{-4}
10^{-6}
$\mathcal{H}_\infty$ error
2
6
10
14
18
reduced model order r

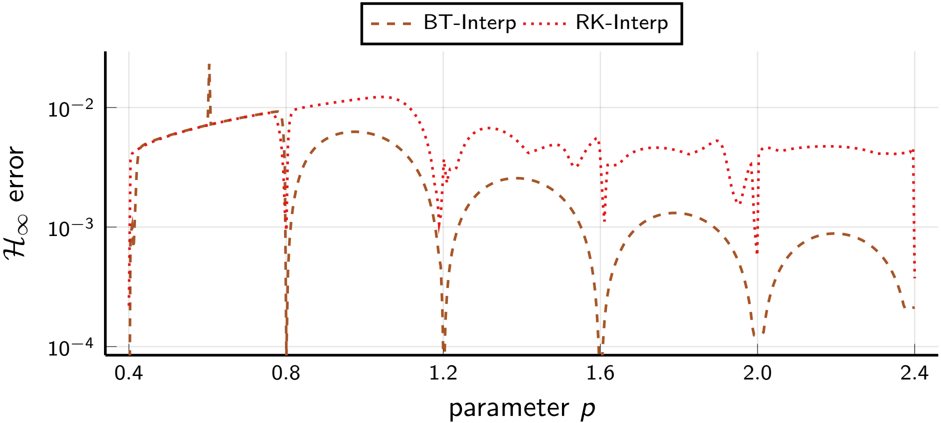
Parametric Systems

$$u(t) \longrightarrow \quad \Sigma(p): \begin{cases} \dot{x} = A(p)x + B(p)u \\ y = C(p)x + D(p)u \end{cases} \quad \longrightarrow y(t)$$

$$A: \Omega \to \mathbb{R}^{n \times n}, \quad B: \Omega \to \mathbb{R}^{n \times m}, \quad C: \Omega \to \mathbb{R}^{p \times n}, \quad D: \Omega \to \mathbb{R}^{p \times m}$$

$$\Omega \subset \mathbb{R}^{n_p}$$

Previous Results (Geuss et al. 2013)



Parametric SOBMOR

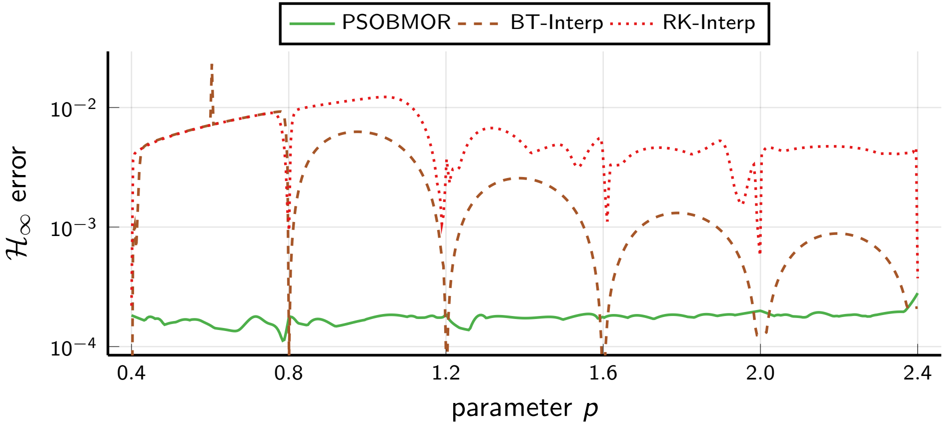
$$\Sigma(\theta, p) : \begin{cases} \dot{x} = (\mathcal{J}(\theta, p) - \mathcal{R}(\theta, p))x + \mathcal{B}(\theta, p) \\ y = \mathcal{C}(\theta, p)x + \mathcal{D}(\theta, p)u \end{cases}$$

$$\mathcal{R}(\theta, p) = g_1(p)R_1(\theta) + \cdots + g_k(p)R_k(\theta)$$

Adapted Objective Function

$$L(\gamma, H, \widetilde{H}(\cdot,\cdot,\theta), S) = \frac{1}{\gamma} \sum_{(s_i,p_i)\in S} \left(\left(\| H(s_i, p_i) - \widetilde{H}(s_i, p_i; \theta) \|_2 - \gamma \right)_+ \right)^2,$$

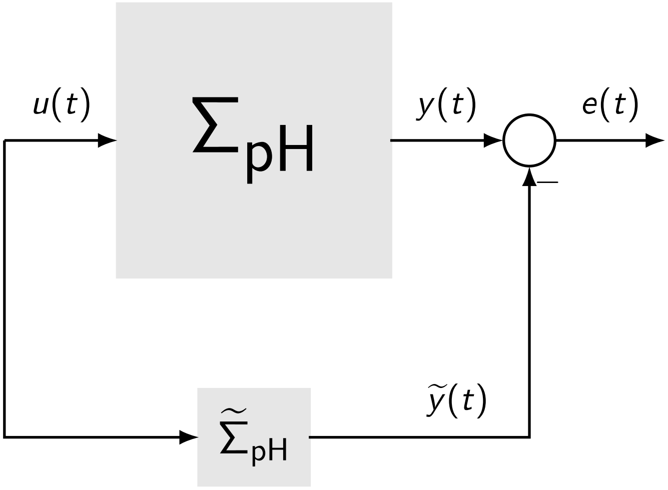
Results


PSOBMOR
BT-Interp
RK-Interp
10^{-2}
10^{-3}
10^{-4}
$\mathcal{H}_\infty$ error
0.4
0.8
1.2
1.6
2.0
2.4
parameter p

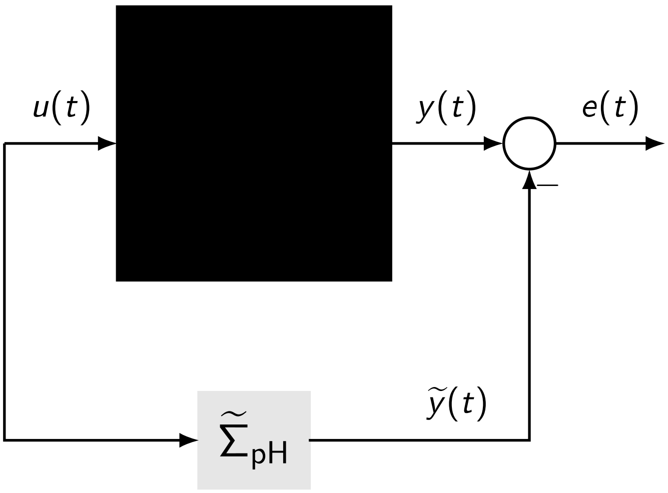
Structure Preserving PMOR

$$\Sigma_{\mathrm{pH}}(\theta, p) : \begin{cases} \dot{x} = (\mathcal{J}(\theta, p) - \mathcal{R}(\theta, p))x + \mathcal{B}(\theta, p)u \\ y(t, p) = \mathcal{B}(\theta, p)^{\mathsf{T}} x \end{cases}$$

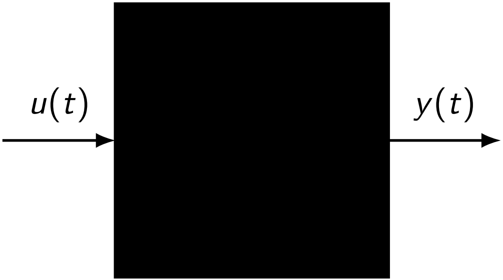
Identification



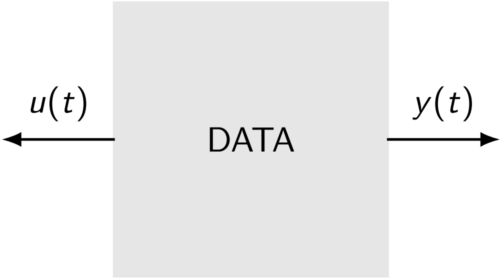
Identification



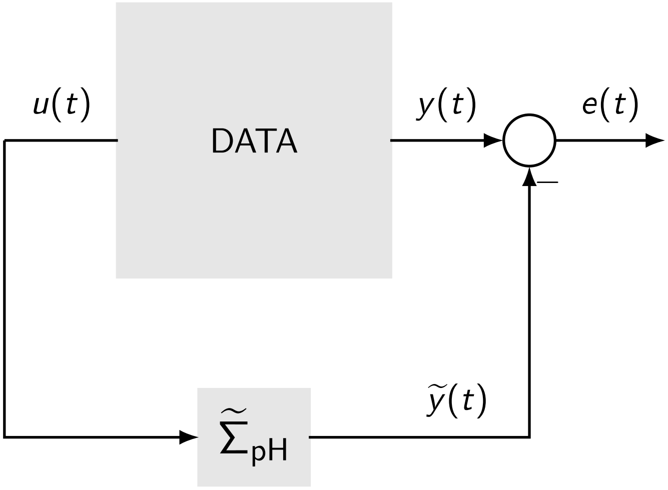
Identification



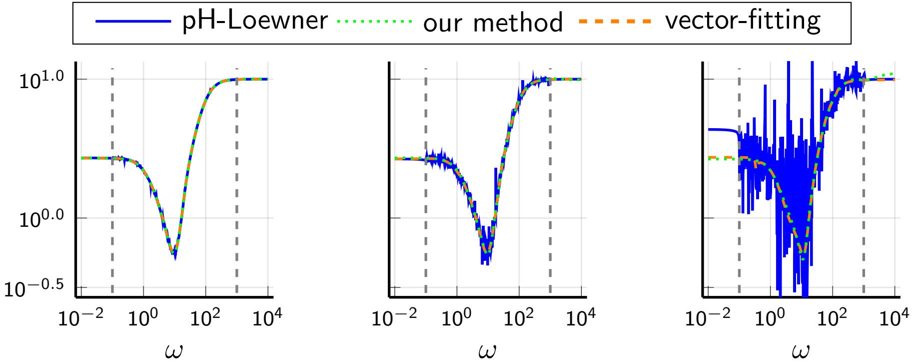
Identification



Identification

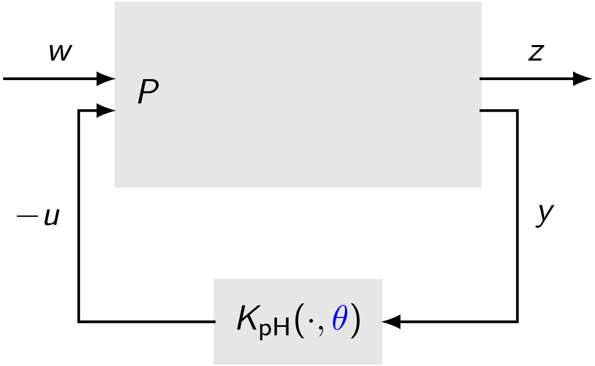


Results: Structured System Identification

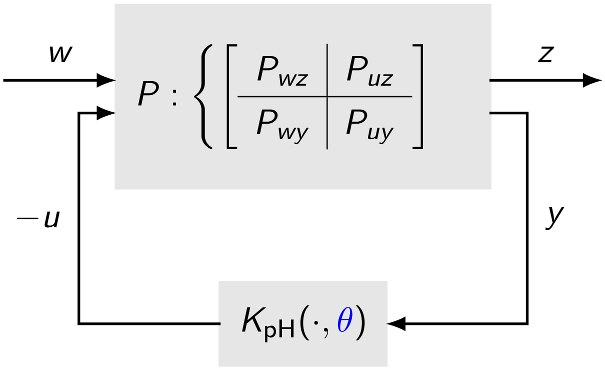


(a) $\sigma = 0.01$ (b) $\sigma = 0.1$ (c) $\sigma = 1.0$

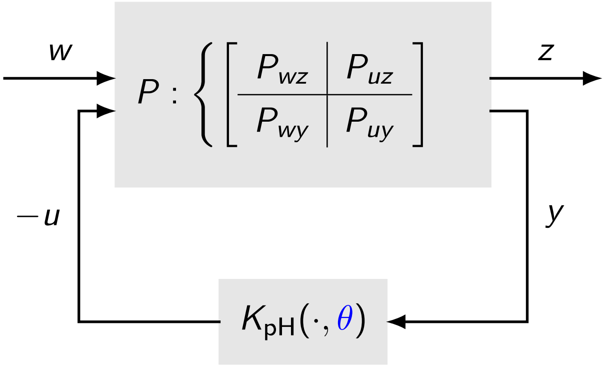
Controller Synthesis



Controller Synthesis



Controller Synthesis



$$P_{CL}(s, \theta) := P_{wz}(s) - P_{uz}(s) K_{\mathrm{pH}}(s, \theta)(I + K_{\mathrm{pH}}(s, \theta) P_{uy}(s))^{-1} P_{wy}(s)$$

$\mathcal{H}_\infty$ Synthesis

Objective Function

$$L(\gamma, H, \widetilde{H}(\cdot,\theta), S) = \frac{1}{\gamma} \sum_{s_i \in S} \left(\left(\|H(s_i) - \widetilde{H}(s_i,\theta)\|_2 - \gamma \right)_+ \right)^2,$$

where $S \subset \mathrm{i}\mathbb{R}$, $\gamma \in \mathbb{R}^+$, and

$$(\,\cdot\,)_+ : \mathbb{R} \to \mathbb{R}^+, x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$\mathcal{H}_\infty$ Synthesis

Objective Function

$$L(\gamma, H, \widetilde{H}(\cdot,\theta), S) = \frac{1}{\gamma} \sum_{s_i \in S} \left((\|P_{CL}(s,\theta)\|_2 - \gamma)_+ \right)^2,$$

where $S \subset \mathrm{i}\mathbb{R}$, $\gamma \in \mathbb{R}^+$, and

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$\mathcal{H}_\infty$ Synthesis

Objective Function

$$L(\gamma, H, \widetilde{H}(\cdot,\theta), S) = \frac{1}{\gamma} \sum_{s_i \in S} \left((\|P_{CL}(s,\theta)\|_2 - \gamma)_+ \right)^2,$$

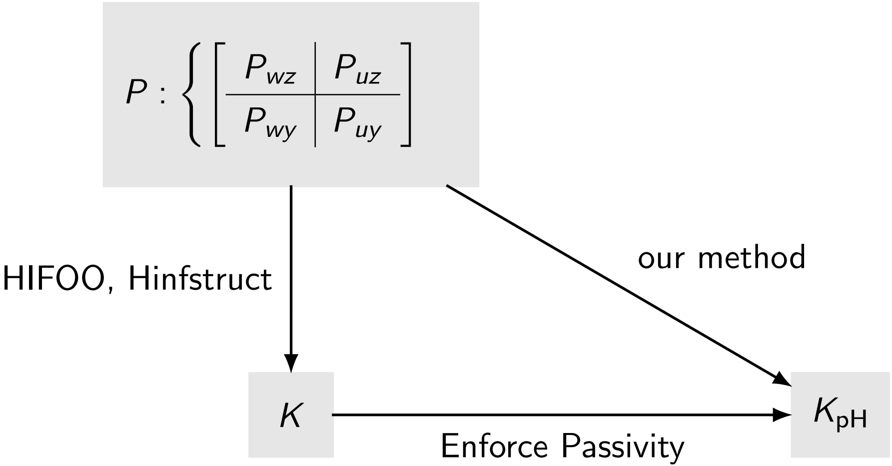
where $S \subset \mathrm{i}\mathbb{R}$, $\gamma \in \mathbb{R}^+$, and

$$(\,\cdot\,)_+ : \mathbb{R} \to \mathbb{R}^+, x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

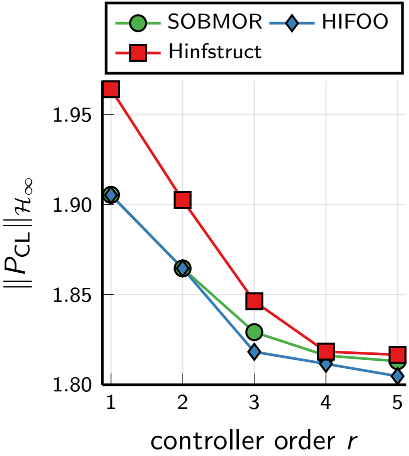
Benefits

- No need to compute the $\mathcal{H}_\infty$-norm of P_{CL}.
- Gradient contains information on all samples s_i where $\|H(s_i) - \widetilde{H}(s_i)\|_2 > \gamma$.

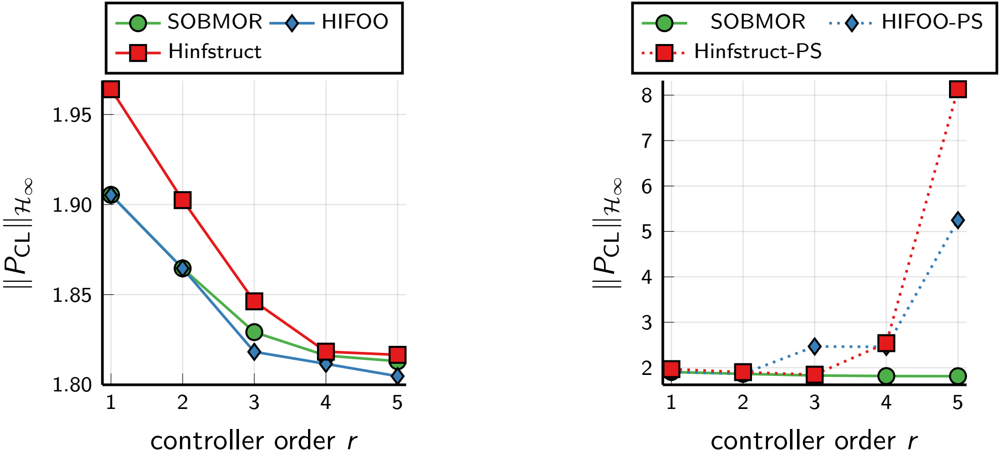
Application



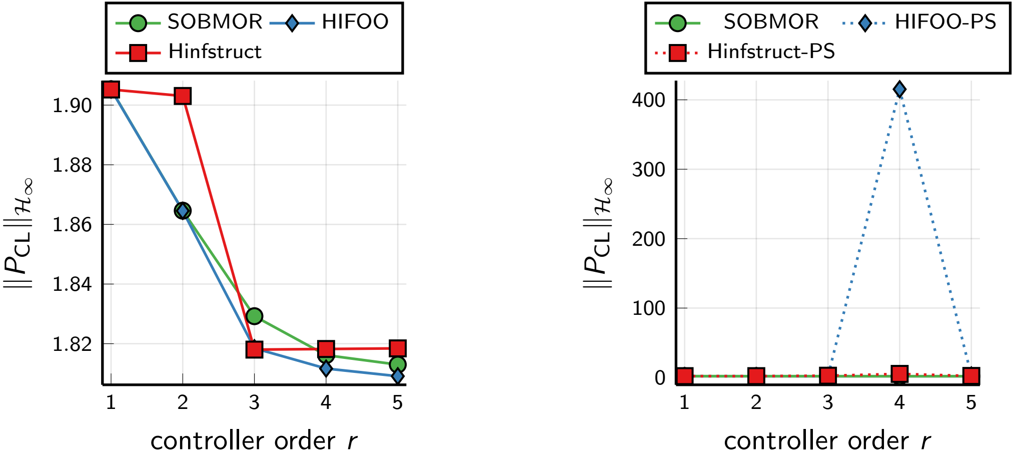
Results — EB5 model



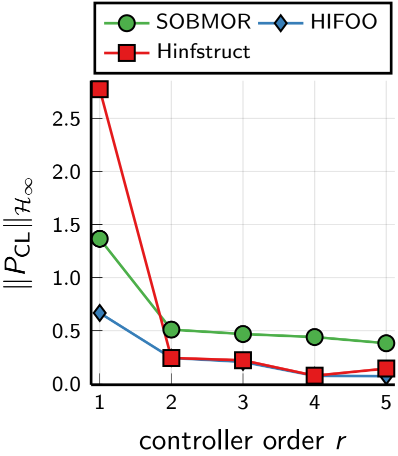
Results — EB5 model

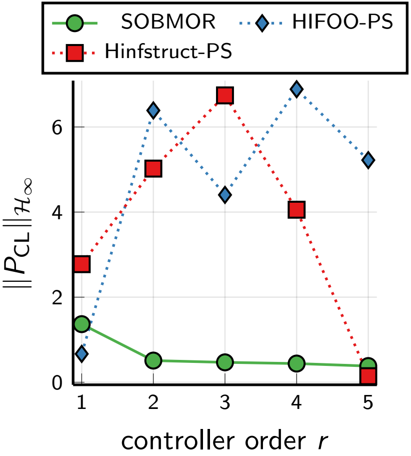


Results — EB6 model



Results — DLR1 model





Collaborators: V. Mehrmann, T. Moser, M. Schaller, M. Voigt

Contribution

- Optimization-based methods for pH systems
- Flexible algorithms for MOR, $\mathcal{H}_\infty$-Synthesis, and Identification
- **Reading**: *Adaptive Sampling for Structure-Preserving Model Order Reduction of Port-Hamiltonian Systems*

Contact

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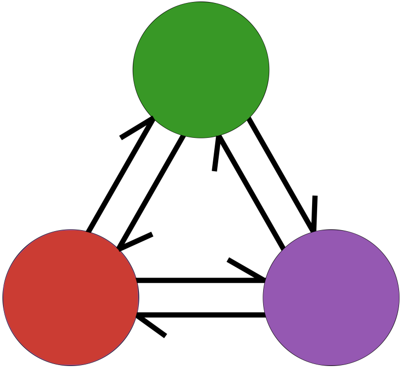

PortHamiltonianBenchmarkCollection

www.port-Hamiltonian.io

Collaborators: V. Mehrmann, T. Moser, M. Schaller, M. Voigt

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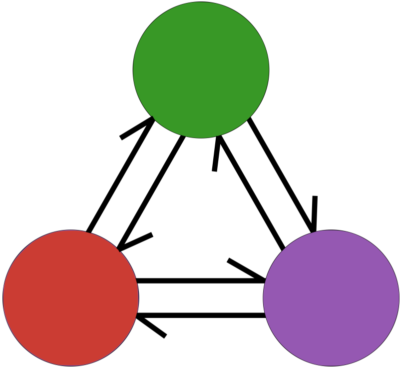

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